

Space and time dimensions of algebras with applications to Lorentzian noncommutative geometry and the standard model

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Abstract

An analogy with real Clifford algebras on even-dimensional vector spaces suggests to assign a space dimension and a time dimension (modulo 8) to an algebra (represented over a complex Hilbert space) containing two self-adjoint involutions and an anti-unitary operator with specific commutation relations.

It is shown that this assignment is compatible with the tensor product, in the sense that a tensor product of such algebras corresponds to the addition of the space and time dimensions. This could provide an interpretation of the presence of such algebras in PT -symmetric Hamiltonians or the description of topological matter.

This construction is used to build the tensor product of Lorentzian (and more generally pseudo-Riemannian) spectral triples, defined over a Krein space. The application to the standard model of particles suggests the identity of the time and space dimensions of the total (manifold+finite algebra) spectral triple. It also suggests the emergence of the pseudo-orthogonal group $SO(4, 6)$ in a grand unified theory.

Key words:

Pseudo-Riemannian manifolds, noncommutative geometry, standard model unification, Clifford algebras

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1. Introduction

Clifford algebras are at the heart of the description of matter not only because fermions (spinors) are their irreducible representations, but also because they classify topological insulators and superconductors [1,2]. They are also used as a template for deeper structures, such as K -theory [3,4] or noncommutative geometry [5], that pervade physics from topological matter to disordered systems and the standard model of particles.

The main aim of this paper is to develop a pseudo-Riemannian analogue of noncommutative geometry, but on the way we put forward a procedure to assign

a space dimension and a time dimension to a class of algebras. What we need is

- A complex Hilbert space \mathcal{H} .
- A self-adjoint involution γ (i.e. $\gamma^2 = 1$) defining a \mathbb{Z}_2 -grading of operators: an operator a on \mathcal{H} is even if $\gamma a \gamma = a$ and odd if $\gamma a \gamma = -a$. For example, γ can be the chirality operator or the inversion symmetry.
- A second self-adjoint involution η , which can be the flat-band Hamiltonian $\text{sign} H$ or a fundamental symmetry.
- A unitary anti-linear map J (i.e. $J^\dagger J = 1$) such that $J^2 = \pm 1$.
- Specific commutation or anticommutation relations between γ , η and J to be described in Eqs. (1) to (4).

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In a Clifford algebra $\mathcal{Cl}(s, t)$ such that $s + t$ is even, these relations determine $s + t \bmod 8$ and $s - t \bmod 8$ in a unique way. We propose to assign the same dimensions $s + t \bmod 8$ and $s - t \bmod 8$ to any algebra satisfying the same relations between γ , η and J . This is similar to the way Atiyah related the KO -dimension to $s - t \bmod 8$ in Clifford algebras [3].

Such an assignment is meaningful because it is compatible with the tensor product: the relations obtained in the (graded) tensor product $A_1 \hat{\otimes} A_2$ of two such algebras correspond to the sum of the space and time dimensions of A_1 and A_2 modulo 8.

When we apply this to a spectral triple of noncommutative geometry, γ is the usual chirality operator, η is a fundamental symmetry defining a Krein-space structure and J is the usual charge conjugation. Our approach allows us to assign a space-time dimension to the finite algebra of the almost-commutative spectral triple of the standard model of particles. By following Barrett's idea [6] we reach the conclusion that the full spectral triple has the same number of space and time dimensions (modulo 8).

The paper starts with a description of γ , η and J in a Clifford algebra, which sets up the correspondence between commutation relations and space-time dimensions. Then, this correspondence is shown to hold for more general algebras by proving that it is compatible with the graded tensor product of algebras. In section 4, we introduce Krein spaces, which is the natural generalization of Hilbert spaces to pseudo-Riemannian manifolds. Section 5 defines the corresponding generalized spectral triples, that we call *indefinite spectral triples*. This framework is then applied to define the spectral triple of the Lorentzian standard model.

2. Automorphisms of Clifford algebras

We investigate the commutation relations of three operators in Clifford algebras over *even-dimensional* vector spaces. Let $\mathcal{Cl}(s, t)$ be the real Clifford algebra generated by the gamma matrices $\gamma_1, \dots, \gamma_{s+t}$ such that γ_i anticommutes with γ_j if $i \neq j$, $\gamma_i^2 = -1$ for $i = 1, \dots, t$ and $\gamma_i^2 = 1$ for $i = t + 1, \dots, s + t$. Since we assume that $d = s + t$ is even, the dimension of the irreducible spinor representation S of this Clifford algebra is $2^{d/2}$.

First, we consider the chirality $\gamma = i^{(s-t)/2} \gamma_1 \dots \gamma_d$, which satisfies $\gamma^2 = 1$ and $\gamma^\dagger = \gamma$. It implements the main automorphism of the Clifford algebra in

$\begin{smallmatrix} n \\ m \end{smallmatrix}$	0	2	4	6
0	(1,1,1,1)	(-1,-1,1,-1)	(-1,1,1,1)	(1,-1,1,-1)
2	(1,1,-1,-1)	(-1,-1,-1,1)	(-1,1,-1,-1)	(1,-1,-1,1)
4	(1,1,-1,1)	(-1,-1,-1,-1)	(-1,1,-1,1)	(1,-1,-1,-1)
6	(1,1,1,-1)	(-1,-1,1,1)	(-1,1,1,-1)	(1,-1,1,1)

Table 1

Signs $(\epsilon, \epsilon'', \kappa, \kappa')$ in terms of $n = s - t \bmod 8$ and $m = s + t \bmod 8$.

$\begin{smallmatrix} n \\ m \end{smallmatrix}$	0	2	4	6
0	(0,0) (4,4)	(1,7) (5,3)	(2,6) (6,2)	(3,5) (7,1)
2	(1,1) (5,5)	(2,0) (6,4)	(3,7) (7,3)	(0,2) (4,6)
4	(2,2) (6,6)	(3,1) (7,5)	(4,0) (0,4)	(1,3) (5,7)
6	(3,3) (7,7)	(4,2) (0,6)	(5,1) (1,5)	(6,0) (2,4)

Table 2

$(s \bmod 8, t \bmod 8)$ in terms of $n = s - t \bmod 8$ and $m = s + t \bmod 8$.

the sense that $\gamma u \gamma = u$ (resp. $\gamma u \gamma = -u$) if u is the product of an even (resp. odd) number of gamma matrices.

Second, we define the fundamental symmetry to be $\eta = i^{(t+1)/2} \gamma_1 \dots \gamma_t$ if t and s are odd, while $\eta = i^{s/2} \gamma_{t+1} \dots \gamma_d$ if t and s are even [7] (see also Ref. [8]). It satisfies $\eta^\dagger = \eta$, $\eta^2 = 1$ and $\gamma_i = \eta \gamma_i^\dagger \eta$. It implements the reversion anti-automorphism (i.e. $\gamma_{i_1} \dots \gamma_{i_k} \mapsto \gamma_{i_k} \dots \gamma_{i_1}$) by $u \mapsto \eta u^\dagger \eta$.

Third, we define a charge conjugation J as follows. In the complexification $\mathbb{C}\ell(d)$ of $\mathcal{Cl}(s, t)$, we can define a canonical real structure c by $c(\lambda u) = \bar{\lambda} u$, where $u \in \mathcal{Cl}(s, t)$ and $\bar{\lambda}$ is the complex conjugate of the complex number λ . There is also an antilinear operator K on S such that $c(u) = K u K^{-1}$. The charge conjugation is the operator defined by $J = \gamma K$. It is anti-unitary (i.e. J is anti-linear and $J^\dagger J = 1$). There is a unique determination of K and J (up to a phase) such that

$$J^2 = \epsilon, \quad (1)$$

$$J \gamma = \epsilon'' \gamma J, \quad (2)$$

$$J \eta = \epsilon \kappa \eta J, \quad (3)$$

$$\eta \gamma = \kappa' \gamma \eta, \quad (4)$$

where the signs $(\epsilon, \epsilon'', \kappa, \kappa')$ are given in terms of $n = s - t \bmod 8$ and $m = s + t \bmod 8$ in Table 1.

The signs can be expressed in terms of the dimensions (m, n) by $\epsilon = (-1)^{n(n+2)/8}$, $\epsilon'' = (-1)^{n/2}$, $\kappa = (-1)^{m(m+2)/8}$ and $\kappa' = (-1)^{(m+n)/2}$.

The charge conjugation operator J was defined so that ϵ and ϵ'' agree with Connes' KO -dimension

tables [5,9]. Related tables can be found in the literature [10–12].

By inverting the relation between (s, t) and (m, n) , we can associate *two* pairs of space and time dimensions (j, k) modulo 8 to every pair (m, n) . Indeed, if (j, k) is a solution of $j - k = n \pmod 8$ and $j + k = m \pmod 8$, then $(j + 4, k + 4)$ is also a solution. This corresponds to the Clifford algebra isomorphism $Cl(s, t + 8) = Cl(s + 8, t) = Cl(s + 4, t + 4)$ [13]. The results are gathered in Table 2. Note that η commutes with γ and J if and only if $m = n$, i.e. $s = 0 \pmod 2$ and $t = 0 \pmod 4$.

More formally, Table 1 defines a bijection ψ between the set $\Sigma = (\mathbb{Z}/2)^4$ of quadruple of signs and the additive subgroup $(2\mathbb{Z}/8) \times (2\mathbb{Z}/8)$ which contains the couples (m, n) . This subgroup is clearly isomorphic to $(\mathbb{Z}/4) \times (\mathbb{Z}/4)$, the isomorphism being $(m, n) \mapsto (\frac{m}{2} \pmod 4; \frac{n}{2} \pmod 4)$. Table 2 also describes an isomorphism. To understand it, let us call H the subgroup of $(\mathbb{Z}/8) \times (\mathbb{Z}/8)$ generated by the element $(4, 4)$. Then the entries of Table 2 are cosets lying in $\frac{(\mathbb{Z}/8) \times (\mathbb{Z}/8)}{H}$. There are 16 such cosets which can be written in a unique way in the form $k\alpha + k'\alpha'$ where α is the coset containing $(1, 1)$, α' is the coset containing $(1, -1)$, and k, k' are integers mod 4. Let us call $G \simeq (\mathbb{Z}/4) \times (\mathbb{Z}/4)$ the group they form. Then Table 2 exactly describes the group isomorphism $\theta : (2\mathbb{Z}/8) \times (2\mathbb{Z}/8) \rightarrow G$ given in formula by $(m, n) \mapsto \frac{m}{2}\alpha + \frac{n}{2}\alpha'$, where $m/2$ and $n/2$ are well-defined mod 4. In the sequel we will write ϕ for the bijection $\theta \circ \psi$ which directly associates a quadruple of signs to the corresponding element in G .

It would seem that γ and η could be interchanged by exchanging ϵ'' and $\epsilon\kappa$. However, this is not really the case because γ is used to define the parity (grading) of an operator acting on the spinor space S (in particular of an element of the Clifford algebra) and this grading will determine the graded tensor product of algebras.

3. Generalization

We generalize the previous results by defining a *mod-8-spacetime representation* to be a quadruple $S = (\mathcal{H}, \gamma, \eta, J)$, where \mathcal{H} is a complex Hilbert space equipped with two self-adjoint involutions γ and η (i.e. $\gamma^2 = \eta^2 = 1$) and an anti-unitary operator J that satisfy Eqs. (1) to (4) for some signs $\epsilon, \epsilon'', \kappa$ and κ' . We denote $\sigma(S) = (\epsilon, \epsilon'', \kappa, \kappa')$. The map ϕ associates space and time dimensions

to each mod-8-spacetime representation. However, as for the Brauer-Wall group [14], this assignment can only be meaningful if it is compatible with the graded tensor product that we define now.

By using the chirality operator γ , we can write $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\gamma v = \pm v$ for $v \in \mathcal{H}_\pm$. An element v of \mathcal{H}_\pm is said to be homogeneous and its parity is $|v| = 0$ if $v \in \mathcal{H}_+$ and $|v| = 1$ if $v \in \mathcal{H}_-$. The parity of a linear or antilinear map T on \mathcal{H} is $|T| = 0$ if $\gamma T \gamma = T$ and $|T| = 1$ if $\gamma T \gamma = -T$. From relations (2) and (4) we see that $\epsilon'' = (-1)^{|J|}$ and $\kappa' = (-1)^{|\eta|}$.

The graded tensor product $\hat{\otimes}$ of operators is defined by $(T_1 \hat{\otimes} T_2)(\phi_1 \otimes \phi_2) = (-1)^{|T_2||\phi_1|} T_1 \phi_1 \otimes T_2 \phi_2$ when ϕ_1 and ϕ_2 are homogeneous. It is the natural tensor product of Clifford algebra theory thanks to Chevalley's relation [15]:

$$Cl(s_1, t_1) \hat{\otimes} Cl(s_2, t_2) = Cl(s_1 + s_2, t_1 + t_2),$$

which shows that the graded tensor product is indeed compatible with space and time dimensions.

Let us consider two mod-8-spacetime representations $S_1 = (\mathcal{H}_1, \gamma_1, \eta_1, J_1)$ and $S_2 = (\mathcal{H}_2, \gamma_2, \eta_2, J_2)$ with signs determined by (m_1, n_1) and (m_2, n_2) , respectively. Then, the graded tensor product $S = S_1 \hat{\otimes} S_2$ is the mod-8-spacetime representation defined by the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and the operators

$$\begin{aligned} \gamma &= \gamma_1 \hat{\otimes} \gamma_2, \\ J &= \gamma_1^{|J_2|} J_1 \hat{\otimes} \gamma_2^{|J_1|} J_2 = J_1 \gamma_1^{|J_2|} \hat{\otimes} J_2 \gamma_2^{|J_1|}, \\ \eta &= i^{|n_1||\eta_2|} \gamma_1^{|n_2|} \eta_1 \hat{\otimes} \gamma_2^{|n_1|} \eta_2 = i^{|n_1||\eta_2|} \eta_1 \gamma_1^{|n_2|} \hat{\otimes} \eta_2 \gamma_2^{|n_1|}. \end{aligned}$$

The peculiar form of η is due to its interpretation as a fundamental symmetry, which will be discussed in section 5. Then, it can be checked that γ and η are self-adjoint involutions and J is an anti-unitary map, which satisfy Eqs (1) to (4) for the signs of some dimensions (m, n) . Indeed, we first observe through an explicit calculation that the signs associated to $S_1 \hat{\otimes} S_2$ only depend on the signs associated to S_1 and S_2 : $\epsilon = (-1)^{|J_1||J_2|} \epsilon_1 \epsilon_2$ (where $(-1)^{|J_1||J_2|}$ depends only on ϵ_1'' and ϵ_2'' because $(-1)^{|J_1||J_2|} = (1 + \epsilon_1'' + \epsilon_2'' - \epsilon_1'' \epsilon_2'')/2$), $\epsilon'' = \epsilon_1' \epsilon_2''$, $\kappa = (-1)^{(|n_1|+|J_1|)(|n_2|+|J_2|)} \kappa_1 \kappa_2$ and $\kappa' = \kappa_1' \kappa_2'$.

Now let A_1 and A_2 be two Clifford algebras such that $\sigma(A_i) = \sigma(S_i)$, $i = 1, 2$. Then $\phi(\sigma(S_1 \otimes S_2)) = \phi(\sigma(A_1 \otimes A_2))$ since $\sigma(S_1 \otimes S_2)$ only depends on $\sigma(S_1) = \sigma(A_1)$ and $\sigma(S_2) = \sigma(A_2)$. However the space and time dimensions are additive for Clifford algebras, hence $\phi(\sigma(A_1 \otimes A_2)) = \phi(\sigma(A_1)) +$

$\phi(\sigma(A_2)) = \phi(\sigma(S_1)) + \phi(\sigma(S_2))$. Thus this additive property extends to all mod-8-spacetime representations as $m = m_1 + m_2 \pmod 8$ and $n = n_1 + n_2 \pmod 8$. Moreover, this tensor product is associative and symmetric.

Extending this classification to odd-dimensional spaces appears non-trivial, let alone because the main automorphism of an odd-dimensional Clifford algebra is not inner [11] (in other words, there is no γ).

These space and time dimensions can be used to classify topological insulators and superconductors with symmetries, as well as for the investigation of PT -symmetric Hamiltonians, because of the presence of a Krein-space structure to which we now turn [16–18], because it is crucial to generalize spectral triples to pseudo-Riemannian manifolds.

4. Krein spaces

It was noticed by Helga Baum [19] that the spinor bundle of a pseudo-Riemannian manifold is naturally equipped with the structure of a Krein space. This was generalized by Alexander Strohmaier to noncommutative geometry [20]. Before introducing Krein spaces, we define a *Hermitian form* on a complex vector space as a sesquilinear form which satisfies the same properties as a scalar product, except for positive-definiteness which is replaced by non-degeneracy: if $(x, y) = 0$ for all y , then $x = 0$.

Substituting a Hermitian form for a scalar product has a striking physical consequence: the possible existence of states with negative norms. These states were first met in physics by Dirac in 1942 in his quantization of electrodynamics [21]. He interpreted negative-norm states as describing a *hypothetical world* [22]. Negative-norm states have now become familiar in physics through their role in the Gupta-Bleuler and Becchi-Rouet-Stora-Tyutin (BRST) quantizations of gauge fields.

Dirac's approach was elaborated by Pauli [23], who introduced an operator η such that $\langle x, y \rangle = (x, \eta y)$ is a scalar product. A vector space with a Hermitian form and such an operator η is called a *Krein space* and η is called a *fundamental symmetry* in mathematics. In most applications, the Hermitian product (\cdot, \cdot) is natural and the scalar product $\langle \cdot, \cdot \rangle$ coming from η is somewhat arbitrary. In a Lorentzian manifold, the scalar product corresponds to the Wick rotation following some choice of a time-like direction (see Ref. [24] for a precise definition).

Krein spaces are a natural framework for gauge field theories [25–27] and Lorentzian spectral triples [20, 24, 28–31].

We present now some essential properties of operators on Krein spaces but, true to the physics tradition, we do not describe their functional analytic properties. If \mathcal{K} is a Krein space, a linear operator $T : \mathcal{K} \rightarrow \mathcal{K}$ has a Krein-adjoint T^\times defined by $(T^\times x, y) = (x, Ty)$ for every x and y in \mathcal{K} . A linear operator is Krein-self-adjoint if $T^\times = T$ and Krein-unitary if $T^\times T = TT^\times = 1$.

An anti-linear map (i.e. a map $T : \mathcal{K} \rightarrow \mathcal{K}$ such that $T(\alpha x + \beta y) = \overline{\alpha}Tx + \overline{\beta}Ty$) has a Krein-adjoint T^\times defined by $(y, T^\times x) = (x, Ty)$. It is Krein-anti-unitary if, furthermore, $T^\times T = TT^\times = 1$.

Any fundamental symmetry η is Krein self-adjoint. It enables us to define the adjoint of T with respect to the scalar product: $T^\dagger = \eta T^\times \eta$. Note that η is self-adjoint if $\eta^2 = 1$. In physical applications the Krein adjoint is the most natural. For example, the Dirac operator on a pseudo-Riemannian manifold is Krein-self-adjoint. In Gupta-Bleuler quantization the Krein adjoint is covariant. In gauge field theory, the BRST charge is Krein self-adjoint. The Hilbert adjoint depends on the choice of a fundamental symmetry, it is not covariant in Gupta-Bleuler quantization and the BRST approach.

We can now define an indefinite spectral triple.

5. Indefinite spectral triple

There are many papers dealing with the extension of noncommutative geometry to Lorentzian geometry [6, 20, 24, 28–51]. Inspired by these references, we define an even-dimensional real indefinite spectral triple to be:

- (i) A $*$ -algebra \mathcal{A} represented on a Krein space \mathcal{K} equipped with a Hermitian form (\cdot, \cdot) and a fundamental symmetry η such that $\eta^2 = 1$. We assume that the representation satisfies $\pi(a^*) = \pi(a)^\times$.
- (ii) A chirality operator γ , i.e. a linear map on \mathcal{K} such that $\gamma^2 = 1$ and $\gamma^\dagger = \gamma$ (where the adjoint γ^\dagger is defined by $\gamma^\dagger = \eta \gamma^\times \eta$). The algebra commutes with γ .
- (iii) An antilinear charge conjugation J , such that $J^\dagger J = 1$.
- (iv) A set of signs $(\epsilon, \epsilon'', \kappa, \kappa')$ describing relations (1) to (4) between γ , η and J .
- (v) A Dirac operator D , which is Krein-self-adjoint and satisfies $JD = DJ$ and $\gamma D =$

$$-D\gamma.$$

We refer the reader to the above references for the functional analytic aspects of indefinite spectral triples. If we compare with Connes' spectral triples, we see that we have an additional object (the fundamental symmetry η) and two additional signs: κ and κ' . Because of this more complex structure, the KO -dimension $n = s - t \pmod 8$ is no longer enough to classify indefinite spectral triples and we need both m and n to classify them. Although a Clifford algebra is not an indefinite spectral triple, the classification carried out in section 3 holds also for indefinite spectral triples.

More precisely, Let $(\mathcal{A}_1, \mathcal{K}_1, D_1, J_1, \gamma_1, \eta_1)$ and $(\mathcal{A}_2, \mathcal{K}_2, D_2, J_2, \gamma_2, \eta_2)$ be two real even-dimensional indefinite spectral triples. Supplement the tensor products defined in section 3 with

$$\begin{aligned}\mathcal{A} &= \mathcal{A}_1 \hat{\otimes} \mathcal{A}_2, \\ \mathcal{K} &= \mathcal{K}_1 \otimes \mathcal{K}_2, \\ D &= D_1 \hat{\otimes} 1 + 1 \hat{\otimes} D_2, \\ \pi &= \pi_1 \hat{\otimes} \pi_2.\end{aligned}$$

To define a Hermitian form (\cdot, \cdot) on \mathcal{K} in terms of the Hermitian forms $(\cdot, \cdot)_1$ and $(\cdot, \cdot)_2$ on \mathcal{K}_1 and \mathcal{K}_2 , we consider the example where \mathcal{K}_1 and \mathcal{K}_2 are the spinor spaces of two Clifford algebras. Then, we use Robinson's theorem [52], which states that there is a unique Hermitian form (up to a real scalar factor) on the space of spinors such that the γ^μ matrices generating the Clifford algebra are Krein-self-adjoint. If we impose that $\gamma^\mu \hat{\otimes} 1$ and $1 \hat{\otimes} \gamma^\nu$ are Krein-self-adjoint for every generator γ^μ (resp. γ^ν) of the first (resp. the second) Clifford algebra, we obtain the following Hermitian form on the tensor product:

$$(\phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2) = i^{|\eta_1||\eta_2|} (\phi_1, \psi_1)_1 (\phi_2, \gamma_2^{|\eta_1|} \psi_2)_2.$$

Since this definition depends only on $|\eta_1|$, $|\eta_2|$ and γ_2 , we can extend it to any mod-8-spacetime representation. The formula for η in the tensor product of two mod-8-spacetime representations is due to the compatibility of the fundamental symmetry with this Hermitian form. This ensures

$$\langle \phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2 \rangle = \langle \phi_1, \psi_1 \rangle_1 \langle \phi_2, \psi_2 \rangle_2,$$

and implies the Kasparov identities [53]

$$\begin{aligned}(T_1 \hat{\otimes} T_2)^\times &= (-1)^{|T_1||T_2|} T_1^\times \hat{\otimes} T_2^\times, \\ (T_1 \hat{\otimes} T_2)^\dagger &= (-1)^{|T_1||T_2|} T_1^\dagger \hat{\otimes} T_2^\dagger,\end{aligned}$$

for the tensor product of two linear operators and

$$\begin{aligned}(T_1 \hat{\otimes} T_2)^\times &= (-1)^{|\eta_1||\eta_2|+|T_1||T_2|} T_1^\times \hat{\otimes} T_2^\times, \\ (T_1 \hat{\otimes} T_2)^\dagger &= (-1)^{|T_1||T_2|} T_1^\dagger \hat{\otimes} T_2^\dagger,\end{aligned}$$

for the tensor product of two antilinear operators.

It can be checked that this tensor product is indeed a real even-dimensional indefinite spectral triple (i.e. $D^\times = D$ and D commutes with J and anticommutes with γ).

The extension of this tensor product to odd-dimensional indefinite spectral triples seems difficult, if one considers the complexity of the Riemannian case [54–58]. Note that Farnsworth also advocates the use of a graded tensor product [58].

6. Lorentzian Standard Model

The fermionic Lagrangian of the Lorentzian Standard Model was obtained by Koen van den Dungen who defined Lorentzian spectral triples similar to ours [31], although the tensor product is not treated in detail and the classification is not carried out.

The noncommutative version of the standard model that we consider includes right-handed neutrinos, which can explain the observed neutrino masses via the seesaw mechanism and the baryon asymmetry of the universe via leptogenesis [59].

Barrett pointed out that, to get rid of the fermion quadrupling problem, the total KO -dimension n of the spectral triple of the standard model has to be zero [6]. Within our classification, this means that the total space dimension of the model is the same as its total time dimension (modulo 8). Since our spacetime corresponds to $(s, t) = (3, 1)$, we have $n_1 = 3 - 1 = 2$ and the KO -dimension of the finite part has to be $n_F = 0 - 2 = 6 \pmod 8$, which is indeed the correct dimension to get a seesaw mechanism [6].

In the spectral triple proposed by Dungen:

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_F & 0 \\ 0 & -\gamma_F \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} c,$$

where c stands for complex conjugation. Note that Dungen's finite spectral triple is not standard because J and D do not commute, but our classification still holds in that case. The signs are computed to be $(\epsilon, \epsilon'', \kappa, \kappa')_F = (1, -1, -1, 1)$. Thus, $(m_F, n_F) = (2, 6)$ and $(s_F, t_F) = (0, 2)$ or $(4, 6)$ modulo 8. As a consequence, the total space-time dimensions of the standard model are $(s, t) = (3, 3)$ or $(7, 7)$ modulo 8.

The result $s = 7$, $t = 7$ has a nice physical interpretation because it corresponds to the Clifford algebra $C\ell(7, 7) = C\ell(3, 1) \hat{\otimes} C\ell(4, 6)$. Since $C\ell(3, 1)$ describes the Lorentz manifold, $C\ell(4, 6)$ could be relevant for the finite part of the standard model algebra. Indeed, it was argued that $SO(4, 6)$, an invariance group of $C\ell(4, 6)$, is as good as $SO(10)$ as a grand unification group, because the chiral spin representations of $SO(10)$ and $SO(4, 6)$ both transform as $\mathbf{16} = (\mathbf{2}, \mathbf{4}) + (\mathbf{2}', \mathbf{\bar{4}})$ and $\mathbf{\bar{16}} = (\mathbf{2}, \mathbf{\bar{4}}) + (\mathbf{2}', \mathbf{4})$ under the regular subgroup $SO(4) \times SO(6)$ [60]. As is well known, this generates the physical representations of the standard model. In other words, our framework suggests that the correct grand unification group could be $SO(4, 6)$ instead of $SO(10)$.

On partly aesthetical ground, Maraner already observed that spacetime coupled to matter might have an equal number of space and time dimensions [60]. In Maraner's approach, several signatures are possible: $(s, t) = (13, 1)$, $(9, 5)$ or $(7, 7)$. Our results enable us to select the group $SO(7, 7)$. As a side remark, $SO(7, 7)$ plays a role in string theory [61] as an invariance group of $G_2/U(2)$ [62].

7. Conclusion

A particularly appealing aspect of noncommutative geometry is that the internal (fibre) and external (manifold) degrees of freedom are put into a common geometric framework. Real Clifford algebras can also unify spacetimes and finite objects since they describe spinors on pseudo-Riemannian manifolds as well as finite geometries [63]. Therefore, it is not surprising that real Clifford algebras can be used to define the space and time dimensions of an algebra representing (in a generalized sense) a possibly noncommutative spacetime. The present paper is a precise formulation of this idea and the main ingredient of the definition of a time dimension is the fundamental symmetry η which allows for a kind of Wick rotation of spacetime.

The main remaining difficulty in the definition of a noncommutative geometric version of the Lorentzian standard model is the determination of the bosonic spectral action. We would like to suggest a possible approach to this problem. In the case of a compact Riemannian manifold, the spectral action is invariant under the huge group of unitary operators U on the Hilbert space $L^2(M, S) \otimes H_F$, such that $UJ = JU$ and $U\gamma = \gamma U$ [9]. The spectral action is then obtained in terms of a sum of inte-

grals of Lagrangian densities multiplied by powers of the cutoff Λ . We conjecture that the same result can be obtained by looking for the Lagrangian densities that are invariant under the huge group and have a dimension corresponding to a specific power of Λ . The advantage of this second point of view is that it can be immediately adapted to the pseudo-Riemannian and non-compact cases. We defer this question to a forthcoming paper.

For applications to topological insulators, it would be desirable to extend these results to the case of odd-dimensional algebras. We already mentioned that this is not straightforward, but a possible direction might be to establish a connection with the generalized Brauer-Wall groups [64] or Salingaros vee-groups [65–68].

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